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ON A POINT LOAD APPLIED TO A MIXTURE OF INTERACTING CONTINUA*

by

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Summary. The problem of the response of an infinite medium, composed of an interacting mixture of elastic solid and viscous fluid, to a point force impulsively applied is studied.

Using a linearized theory for the mixture, the problem is solved with the aid of integral transforms. Inversion of the solution is accomplished for the case where the diffusive resistance is zero. The displacement field for the solid and the velocity field of the fluid are given in terms of integrals and infinite series.

Under further restrictions these solutions are given in forms involving only delta and step functions.

1. Introduction. The purpose of this paper is to study the response of an infinite body to a stationary impulsive body force when the body is composed of an interacting mixture of solid and fluid.

The linearized mixture theory used is that proposed by Green and Steel [1]* for an isotropic elastic solid and Newtonian fluid. Steel [2] used this theory to study the propagation of plane waves through such a medium while Atkin [3] generalized both [1] and [2] and included thermal effects. In a continuing investigation; uniqueness theorems were discussed by Atkin, Chadwick and Steel [4] and from their results one has available both suitable sets of boundary conditions and sets of inequalities that the material constants must satisfy.

An alternate formulation of boundary value problems for the mixture has been given by Atkin [5] in terms of a decomposition of solid displacement and fluid velocity vectors into the gradient of scalar functions and curl of vector functions. The new formulation leads one to solve coupled systems of partial differential equations for these quantities. In [5], Atkin has also considered the propagation of small amplitude waves when the mixture is restricted to isotropic elastic solid and inviscid fluid.

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*Numbers inside square brackets refer to the references cited in section 8.

The problem discussed in this paper has been studied in classical elasticity by Payton [6] who used the solution to study the motion of an elastic body under the influence of an impulsive moving point force acting in the direction of the line of motion: The connection between the stationary and moving point force problems is the dynamic Betti-Rayleigh reciprocal theorem used in [6] and again by Payton in [7]. Earlier uses of such a theorem are mentioned in [6]. An analogous theorem to the Betti-Rayleigh theorem has been established by the author and the results obtained here are used to study the moving point-force problem for a mixture.

In section 2 we pose the mathematical problem to be studied and in section 3 we solve this problem by means of Fourier and Laplace transforms. Fourier inversion is accomplished in section 4 and in sections 5 and 6 the Laplace transform is inverted to yield integral representations of the solution for the special case when the diffusive force parameter is zero. In the final section, various special cases are discussed.

2. The mathematical problem: The complete system of linearized field equations for a mixture of isotropic elastic solid and viscous fluid, according to [2], consists of the equations of motion

$$\begin{aligned} &\sigma_{ij,j} - \pi_i + \bar{\rho}_1 f_i = \bar{\rho}_1 \ddot{w}_i, \\ &\pi_{ij,j} + \pi_i + \bar{\rho}_2 g_i = \bar{\rho}_2 \dot{v}_i, \qquad i = 1,2,3 , \end{aligned} \tag{2.1}$$

the continuity equations

$$\rho_1 = \bar{\rho}_1(1-e_{kk}) \quad , \quad \dot{\eta} + \bar{\rho}_2 \, v_{k,k} = 0 \quad , \tag{2.2}$$

the strain-displacement equations

$$2e_{ij} = w_{i,j} + w_{j,i} \quad , \quad i,j = 1,2,3 \quad , \tag{2.3a}$$

the rate of deformation-velocity relations,

$$2 \, f_{ij} = v_{i,j} + v_{j,i} \quad , \quad i,j = 1,2,3, \tag{2.3b}$$

and the constitutive equations

$$\left.\begin{aligned} \sigma_{ij} &= \alpha_1 \, \delta_{ij} + 2\beta_3 e_{ij} + \beta_2 \delta_{ij} e_{kk} + \beta_1 \eta \delta_{ij}, \\ \pi_{ij} &= -\alpha_1 \delta_{ij} - \gamma_2 \delta_{ij} e_{kk} + 2\mu f_{ij} + \lambda \delta_{ij} f_{kk} - \gamma_1 \eta \delta_{ij}, \\ \pi_i &= \frac{\bar{\rho}_1 \alpha_1}{\bar{\rho} \, \bar{\rho}_2} \eta_{,i} - \frac{\bar{\rho}_2 \alpha_1}{\bar{\rho}} \, e_{kk,i} + \alpha(\dot{w}_i - v_i) \; , \; i,j = 1,2,3. \end{aligned}\right\} \tag{2.4}$$

These equations are posed for a cartesian coordinate system x_i , $i = 1,2,3$ and for time t assumed to occupy the region

$$-\infty < x_1, \, x_2, \, x_3 < +\infty \quad , \quad t \geq 0. \tag{2.5}$$

Indicial notation has been used with a repeated index indicating a sum over 1,2,3. Partial derivatives with respect to x_i and t are indicated by a subscript comma ,i, and superscript ,dot, respectively.

Quantities w_i, e_{ij}, σ_{ij}, f_i refer to the solid constituent and designate, respectively, components of displacement, strain, partial stress and body force. Initial density of the solid is $\bar{\rho}_1$. Similarly, v_i, f_{ij}, π_{ij} and g_i are fluid components of velocity, rate of deformation, partial stress and body force, while $\bar{\rho}_2$ is the initial fluid density.

In (2.2), ρ_1 is the current value (at time t) of the density in the solid, while η is the current fluid density minus the initial value $\bar{\rho}_2$.

The vector with components π_i is called the diffusive resistance vector and arises due to the interaction of the two constituents. In particular, the parameter α (appearing in (2.4c)) is called the diffusive force parameter by Steel [2].

The material constants appearing in (2.4) are assumed to obey the inequalities posed by Atkin, Chadwick and Steel [4]. These are

$$\begin{gathered}\mu \geq 0\,, \quad \alpha \geq 0, \quad 2\mu + 3\lambda \geq 0\,, \quad \beta_3 \geq 0, \\ \bar{\rho}_2\gamma_1 - \frac{\bar{\rho}_1\alpha_1}{\bar{\rho}} \geq 0, \quad \beta_2 + \frac{2\beta_3}{3} + \frac{\bar{\rho}_2\alpha_1}{\bar{\rho}} \geq 0, \\ (\gamma_2 + \frac{\bar{\rho}_2\alpha_1}{\bar{\rho}})^2 \leq (\beta_2 + \frac{2\beta_3}{3} + \frac{\bar{\rho}_2\alpha_1}{\bar{\rho}})\,(\bar{\rho}_2\,\gamma_1 - \frac{\bar{\rho}_1\alpha_1}{\bar{\rho}}).\end{gathered} \tag{2.6}$$

To complete the formulation of the problem given in (2.1) to (2.5) we next specify the body force, the initial and boundary conditions.

The body force acts on the solid component at a point $\vec{x}_0$ in the direction parallel to the x_1-axis. It is applied impulsively. Hence we specify

$$\vec{f}(x_1, x_2, x_3, t) = \vec{a}_1 \ \delta(\vec{x} - \vec{x}_o)\delta(t)$$
$$\vec{g}(x_1, x_2, x_3, t) = \vec{0} \tag{2.7}$$
$$\vec{x}_o = x_o\vec{a}_1 + y_o\vec{a}_2 + z_o\vec{a}_3$$

where $\vec{a}_j$ is a unit vector in the x_j direction and $\delta(\ )$ is the delta function.

Initial conditions imposed are those of zero displacement and velocity,

$$w_i(x_1, x_2, x_3, 0) = \dot{w}_i(x_1, x_2, x_3, 0) = 0$$
$$v_i(x_1, x_2, x_3, 0) = 0 \quad , \quad i = 1, 2, 3, \tag{2.8}$$

and since the body is of infinite extent, regularity conditions are chosen:

$$w_i(R,t),\ v_i(R,t),\ \sigma_{ij}(R,t),\ \pi_{ij}(R,t) \to 0 \quad \text{as} \quad R \to \infty \tag{2.9}$$

where

$$R = (x_1^2 + x_2^2 + x_3^2)^{1/2}.$$

The method of solution is by integral transforms to be applied to the displacement-velocity formulation of the above. This formulation is obtained by using (2.3) and (2.4) in (2.1) and by specifying (2.7). Doing this we have

$$\beta_3 \ w_{i,jj} + (K_1 - \beta_3) \ w_{k,ki} + \frac{\Theta_1}{\bar{\rho}_2} \eta,_i - \alpha(\dot{w}_1 - v_i) - \bar{\rho}_1\ddot{w}_i =$$
$$- \bar{\rho}_1\delta(\vec{x}-\vec{x}_o)\delta(t)\delta_{i1} \ , \tag{2.10}$$
$$-\Theta_1 \ w_{k,ki} + \mu v_{i,jj} + (K_2-\mu)v_{k,ki} - \frac{\Theta_2}{\bar{\rho}_2} \eta,_i + \alpha(\dot{w}_i - v_i) =$$
$$\bar{\rho}_2\dot{v}_i \ , \quad i = 1, 2, 3.$$

Coefficients introduced into (2.10) are

$$K_1 = \beta_2 + 2\beta_3 + \frac{\bar{\rho}_2\alpha_1}{\bar{\rho}}, \quad K_2 = 2\mu + \lambda,$$
$$\Theta_1 = \bar{\rho}_2\beta_1 - \frac{\bar{\rho}_1\alpha_1}{\bar{\rho}} = \gamma_2 + \frac{\bar{\rho}_2\alpha_1}{\bar{\rho}}, \quad \Theta_2 = \bar{\rho}_2\gamma_1 - \frac{\bar{\rho}_1\alpha_1}{\bar{\rho}} \tag{2.11}$$

as defined by Steel in [2].

The problem is now posed: Find w_i, v_i satisfying (2.10), (2.2), (2.8) and (2.9) subject to (2.6).

To facilitate handling we introduce a translation of origin from $\vec{x}$ to $\vec{x}_o$ by setting

$$\vec{x}' = \vec{x} - \vec{x}_o. \tag{2.12}$$

The only visible effect is to replace $\delta(\vec{x} - \vec{x}_o)$ in (2.10) by $\delta(\vec{x}')$. Consider (2.12) in effect and drop all primes.

Before proceeding to the solution we note that the material coefficients (2.11) have obvious interpretations in the theories of elasticity and viscous fluids. If, for example, the fluid component were absent, then one would identify the Lamé constants of elasticity, μ_E, λ_E as

$$\beta_3 = \mu_E, \quad \beta_2 = \lambda_E, \quad \bar{\rho} = \bar{\rho}_1 \tag{2.13a}$$

and set

$$\eta = \beta_1 = \gamma_2 = \mu = \lambda = \alpha = \Theta_1 = \Theta_2 = \alpha_1 = 0. \tag{2.13b}$$

On the other hand, if the solid were absent, then the viscosities of viscous fluid theory, μ_F, λ_F would be

$$\mu = \mu_F, \quad \lambda = \lambda_F, \quad \bar{\rho} = \bar{\rho}_2 \tag{2.14a}$$

and

$$\bar{\rho}_1 = \gamma_1 = \alpha = \beta_2 = \beta_3 = \beta_1 = \gamma_2 = 0 \tag{2.14b}$$

while α_1 would be the thermodynamic pressure,

3. <u>Application of the transforms and solution in the transform variables</u>. We begin by defining the Fourier and Laplace transforms of a function $f(x_1, x_2, x_3, t)$ to be

$$\hat{f}(\lambda_1, \lambda_2, \lambda_3, p) =$$

$$\frac{1}{(2\pi)^{3/2}} \iiint_{-\infty}^{+\infty} e^{-i\lambda_m x_m} \int_0^{\infty} e^{-pt} f(x_1,x_2,x_3,t)\,dt\, dx_1 dx_2 dx_3. \tag{3.1}$$

The inverse of $\hat{f}(\lambda_1,\lambda_2,\lambda_3,p)$ with respect to any of the λ_j's is, for example

$$\hat{f}(\lambda_1,\lambda_2,x_3,p) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{i\lambda_3 x_3}\, \hat{f}(\lambda_1,\lambda_2,\lambda_3,p)\, d\lambda_3 \tag{3.2a}$$

and with respect to p is

$$\hat{f}(\lambda_1,\lambda_2,\lambda_3,t) = \frac{1}{2\pi i} \int_{\omega-i\infty}^{\omega+i\infty} e^{pt}\, \hat{f}(\lambda_1,\lambda_2,\lambda_3,p)\,dp \tag{3.2b}$$

for $\hat{f}(\lambda_1,\lambda_2,\lambda_3,p)$ regular in the strip $\mathrm{Re}(p) > \omega$.

The complete inverse of $\hat{f}$ is

$$f(x_1,x_2,x_3,p) = \tag{3.2c}$$

$$\frac{1}{(2\pi)^{3/2}} \iiint_{-\infty}^{+\infty} \left\{\frac{1}{2\pi i} \int_{\omega-i\infty}^{\omega+i\infty} e^{i\lambda_j x_j + pt}\, \hat{f}(\lambda_1,\lambda_2,\lambda_3,p)\,dp\right\} d\lambda_1 d\lambda_2 d\lambda_3.$$

In addition to (3.1) and (3.2) we use properties of $\delta(x_k)$ given by

$$\int_{-\infty}^{+\infty} \bar{e}^{i\lambda_s x_s}\,\delta(x_s)\,dx_s = 1 \quad \text{(no sum on s)} \,, \tag{3.3}$$

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{i\lambda_s x_s}\,d\lambda_s = \delta(x_s) \quad \text{(no sum on s, s = 1,2,3)}.$$

Applying (3.1) to (2.10) and the second of (2.2), and using (2.8), (2.9), (2.12) and (3.3) we have immediately that

$$-[\beta_3\lambda_j\lambda_j + \alpha p + \bar{\rho}_1 p^2]\,\hat{w}_m - (K_1-\beta_3)\lambda_m\lambda_j\hat{w}_j + \frac{i}{\bar{\rho}_2}\,\theta_1\lambda_m\hat{\eta} + \alpha\,\hat{v}_m$$

$$= -\frac{\bar{\rho}_1\delta_{m1}}{(2\pi)^{3/2}} \,, \tag{3.4}$$

$$\theta_1\lambda_m\lambda_k\hat{w}_k - [\mu\lambda_j\lambda_j + \alpha + \bar{\rho}_2 p]\hat{v}_m - (K_2-\mu)\lambda_m\lambda_k\hat{v}_k + \alpha p\hat{w}_m - \frac{i\theta_2}{\bar{\rho}_2}\lambda_m\hat{\eta} = 0,$$

$$p\hat{\eta} + i\,\bar{\rho}_2\,\lambda_k\hat{v}_k = 0 \,, \qquad m = 1,2,3. \tag{3.5}$$

First we eliminate the terms $\lambda_j\hat{w}_j$ and $\lambda_j\hat{v}_j$. Do so by substituting (3.5) into (3.4), multiplying by λ_m, and summing on m. This gives

$$[K_1\lambda_j\lambda_j + \alpha p + \bar{\rho}_1 p^2]\lambda_m\hat{w}_m - \frac{[\theta_1\lambda_j\lambda_j + \alpha p]}{p}\,\lambda_m\hat{v}_m = -\frac{\bar{\rho}_1\lambda_1}{(2\pi)^{3/2}},$$

$$[\theta_1\lambda_j\lambda_j + \alpha p]\lambda_m\hat{w}_m - [\ (K_2 + \frac{\theta_2}{p})\lambda_j\lambda_j + \alpha + \bar{\rho}_2 p\,]\lambda_m\hat{v}_m = 0. \tag{3.6}$$

Solutions of (3.6) are

$$\lambda_m\hat{w}_m[\{K_1\lambda_j\lambda_j + \alpha p + \bar{\rho}_1 p^2\}\{[K_2 + \frac{\theta_2}{p}]\lambda_j\lambda_j + \alpha + \bar{\rho}_2 p\} - \frac{1}{p}\{\theta_1\lambda_j\lambda_j + \alpha p\}^2\,]$$

$$= \frac{\bar{\rho}_1\lambda_1}{(2\pi)^{3/2}}\ [\ (K_2 + \frac{\theta_2}{p}\,)\lambda_j\lambda_j + \alpha + \bar{\rho}_2 p], \tag{3.7}$$

and

$$\lambda_m \hat{v}_m [\{K_1\lambda_j\lambda_j + \alpha p + \bar{\rho}_1 p^2\}\{[K_2 + \frac{\Theta_2}{p}]\lambda_j\lambda_j + \alpha + \bar{\rho}_2 p\} - \frac{1}{p}\{\Theta_1\lambda_j\lambda_j + \alpha p\}^2]$$

$$= \frac{\bar{\rho}_1\lambda_1}{(2\pi)^{3/2}} [\Theta_1\lambda_j\lambda_j + \alpha p] \tag{3.8}$$

while $\hat{\eta}$ is known from (3.5).

Now knowing $\lambda_j\hat{w}_j$, $\lambda_j\hat{v}_j$ and $\hat{\eta}$, solve (3.4) directly for $\hat{w}_m$ and $\hat{v}_m$. After a little algebra we may write

$$\hat{w}_m = \frac{\bar{\rho}_1}{(2\pi)^{3/2}} \frac{N_m}{[\Delta_1\Delta_4 - \alpha^2 p][\Delta_2\Delta_3 - \frac{\Delta_5^2}{p}]}, \tag{3.9}$$

$$\hat{v}_m = \frac{\bar{\rho}_1}{(2\pi)^{3/2}} \frac{M_m}{[\Delta_1\Delta_4 - \alpha^2 p][\Delta_2\Delta_3 - \frac{\Delta_5^2}{p}]}, \tag{3.10}$$

where

$$N_m = \Delta_1[\Delta_2\{\delta_{m1}\Delta_3 - (K_1 - \beta_3)\lambda_1\lambda_m\} - \frac{\delta_{m1}}{p}\Delta_5^2 + \frac{\Theta_1}{p}\lambda_1\lambda_m\{\Delta_5 + \alpha p\}]$$

$$- \alpha^2 p\, \lambda_1\lambda_m(K_2 - \mu + \frac{\Theta_2}{p}), \tag{3.11}$$

$$M_m = \Delta_2\, \alpha\, p[\delta_{m1}\Delta_3 - (K_1 - \beta_3)\lambda_1\lambda_m] + \Delta_5\, \alpha[\lambda_1\lambda_m\Theta_1 - \delta_{m1}\Delta_5]$$

$$+ \lambda_1\lambda_m\Delta_4[\Theta_1\Delta_1 - (K_2 + \frac{\Theta_2}{p} - \mu)\,\alpha\, p],\ m = 1,2,3, \tag{3.12}$$

and

$$\left.\begin{aligned} &\Delta_1 = \mu\,\lambda_j\lambda_j + \alpha + \bar{\rho}_2 p\,, \quad \Delta_2 = (K_2 + \frac{\Theta_2}{p})\lambda_j\lambda_j + \alpha + \bar{\rho}_2 p, \\ &\Delta_3 = K_1\lambda_j\lambda_j + \alpha p + \bar{\rho}_1 p^2\,, \quad \Delta_4 = \beta_3\lambda_j\lambda_j + \alpha p + \bar{\rho}_1 p^2\,, \\ &\Delta_5 = \Theta_1\lambda_j\lambda_j + \alpha p. \end{aligned}\right\} \tag{3.13}$$

Equations (3.9), (3.10) constitute the complete solution of the problem in the transform variables.

4. <u>Fourier integral transform inversion</u>. Inversion of (3.9), (3.10) is accomplished in two steps by inverting with respect to λ_1 first and then inverting with respect to λ_2 and λ_3 simultaneously. This process is given in detail for (3.9) but only the final form for (3.10) is written.

We begin by factoring the denominator of (3.9) by writing

$$\Delta_1\Delta_4 - \alpha^2 p = \beta_3\mu[\lambda_j\lambda_j + P_1][\lambda_j\lambda_j + P_2] \tag{4.1}$$
$$\Delta_3\Delta_2 - \frac{\Delta_5^2}{p} = [K_1(K_2 + \frac{\Theta_2}{p}) - \frac{\Theta_1^2}{p}][\lambda_j\lambda_j + P_3][\lambda_j\lambda_j + P_4]$$

where the P_k are given by

$$P_{1,2} = \frac{\epsilon_1 \pm R_1}{2\beta_3\mu}\ , \quad P_{3,4} = \frac{\epsilon_2 \pm R_2}{2[K_1(K_2+\frac{\Theta_2}{p}) - \frac{\Theta_1^2}{p}]}\ , \tag{4.2}^*$$

with

$$\epsilon_1 = \alpha[\beta_3 + p\mu] + \bar{\rho}_2\, p\, \beta_3 + \bar{\rho}_1\, \mu\, p^2, \tag{4.3}$$

$$R_1 = [\{\alpha(\beta_3-\mu p) + \bar{\rho}_2 p\beta_3 - \mu\bar{\rho}_1 p^2\}^2 + 4\beta_3\mu\alpha^2 p]^{1/2} \tag{4.4}$$

$$\epsilon_2 = \alpha[K_1 + p(K_2 + \frac{\Theta_2}{p}) - 2\Theta_1] + \bar{\rho}_2 pK_1 + \bar{\rho}_1 p^2(K_2 + \frac{\Theta_2}{p}), \tag{4.5}$$

$$R_2 = [\{\alpha[K_1 - p(K_2 + \frac{\Theta_2}{p})] + \bar{\rho}_2 pK_1 - \bar{\rho}_1 p^2(K_2 + \frac{\Theta_2}{p})\}^2$$
$$+ 4\{\alpha(\Theta_1-K_1) + \bar{\rho}_1 p\Theta_1\}\{\alpha(\Theta_1 - [K_2 + \frac{\Theta_2}{p}]) + \bar{\rho}_2 p\,\Theta_1\}]^{1/2}. \tag{4.6}$$

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*In $P_{k,\ell}$ associate the (+) sign with k, the (−) sign with ℓ.

Now incorporate (4.1) into (3.9) and expand (3.9) into fractions in each term of which the numerator is independent of λ_1^2.

Accordingly we obtain,

$$\hat{w}_m(\lambda_1,\lambda_2,\lambda_3,p) = -\frac{\bar{\rho}_1}{(2\pi)^{3/2}}\sum_{k=1}^{4}\frac{\phi_{km}}{\lambda_s\lambda_s + P_k}, \quad m = 1,2,3 \quad (4.7)$$

where

$$\phi_{km} \equiv \frac{N_m[-(\lambda_2^2+\lambda_3^2) - P_k]}{\beta_3\mu[K_1(K_2+\frac{\Theta_2}{p}) - \frac{\Theta_1^2}{p}]\prod_{\substack{s=1\\ s\neq k}}^{4}(P_k - P_s)}, \quad m = 1,2,3$$

or

$$\phi_{km} = \frac{\delta_{m1}n_1(-P_k) + \delta_{m1}(\lambda_2^2+\lambda_3^2)n_2(-P_k) + \lambda_1\lambda_\nu\delta_{m\nu}n_2(-P_k),}{\beta_3\mu[K_1(K_2+\frac{\Theta_2}{p}) - \frac{\Theta_1^2}{p}]\prod_{\substack{s=1\\ s\neq k}}^{4}(P_k - P_s)} \quad (4.8)$$

$$m = 1,2,3, \quad \nu = 2,3$$

if we define n_1, n_2 by

$$n_1(-P_k) = \Delta_1(-P_k)[\Delta_2(-P_k)\{\Delta_3(-P_k) + (K_1-\beta_3)P_k\} - \frac{\Delta_5^2(-P_k)}{p} - \frac{\Theta_1 P_k}{p}\{\Delta_5(-P_k) + \alpha p\}] + \alpha^2 p(K_2 - \mu + \frac{\Theta_2}{p})P_k, \quad (4.9)$$

$$n_2(-P_k) = \Delta_1(-P_k)[(K_1-\beta_3)\Delta_2(-P_k) - \frac{\Theta_1}{p}\{\Delta_5(-P_k) + \alpha p\}] + \alpha^2 p(K_2 - \mu + \frac{\Theta_2}{p}). \quad (4.10)$$

Applying now the Fourier inversion integral to (4.7) (with respect to λ_1 only) and recalling (3.2a) we find

$$\hat{w}_m(x_1,\lambda_2,\lambda_3,p) = -\frac{\bar{\rho}_1}{4\pi}\sum_{k=1}^{4}\left\{\frac{\delta_{m1}[n_1(-P_k) + (\lambda_2^2+\lambda_3^2)n_2(-P_k)}{\beta_3\mu[K_1(K_2+\frac{\Theta_2}{p}) - \frac{\Theta_1^2}{p}]\prod_{\substack{s=1\\ s\neq k}}^{4}(P_k-P_s)}\right. .$$

$$\left. \cdot \frac{\exp\{-|x_1|[\lambda_2^2+\lambda_3^2+P_k]^{1/2}\}}{[\lambda_2^2+\lambda_3^2+P_k]^{1/2}}\right\}$$

$$-\frac{\bar{\rho}_1}{4\pi}\sum_{k=1}^{4}\left\{\frac{i\,\lambda_\nu\,\delta_{m\nu}\,n_2(-P_k)}{\beta_3\mu[K_1(K_2+\frac{\Theta_2}{p}) - \frac{\Theta_1^2}{p}]\prod_{\substack{s=1\\ s\neq k}}^{4}(P_k-P_s)}\right. .$$

$$\left. \cdot \frac{\partial}{\partial x_1}\left[\frac{\exp\{-|x_1|[\lambda_2^2+\lambda_3^2+P_k]^{1/2}\}}{[\lambda_2^2+\lambda_3^2+P_k]^{1/2}}\right]\right\}, \qquad (4.11)$$

$$m = 1,2,3, \qquad \nu = 2,3,$$

provided

$$\mathrm{Re}\,[\lambda_2^2+\lambda_3^2+P_k]^{1/2} > 0, \qquad k = 1,2,3,4. \qquad (4.12)$$

Equation (4.11) was obtained by using [8], p. 8, no. 11.

We turn again to (3.2a) with the object of inverting (4.11) with respect to λ_2 and λ_3. For this purpose we change variables in the inversion integral and set

$$\hat{w}_m(x_1,x_2,x_3,p) = \frac{1}{2\pi}\int_0^\infty\int_\omega^{2\pi+\omega} e^{ir\Omega\cos(\omega-\epsilon)}\,\hat{w}_m(x_1,r,\epsilon,p)\,r\,d\epsilon\,d\omega \qquad (4.13)$$

where

$$x_2 = \Omega\cos\omega, \quad x_3 = \Omega\sin\omega$$
$$\lambda_2 = r\cos\epsilon, \quad \lambda_3 = r\sin\epsilon. \qquad (4.14)$$

Now put (4.11) into (4.13), use (4.14) and the Poisson integral representation for the Bessel function $J_0(\Omega r)$ (see [9] for example) to obtain

$$\hat{w}_m(x_1,x_2,x_3,p) =$$

$$- \frac{\bar{\rho}_1}{4\pi} \sum_{k=1}^{4} \frac{1}{\beta_3\mu[K_1(K_2 + \frac{\theta_2}{p}) - \frac{\theta_1^2}{p}] \prod_{\substack{s=1 \\ s\neq k}}^{4} (P_k - P_s)} \Bigg\{ \tag{4.15}$$

$$\delta_{m1} \int_0^\infty \frac{n_1(-P_k) + r^2 n_2(-P_k)}{[r^2 + P_k]^{1/2}} \quad J_0(\Omega r) \exp\{-|x_1|[r^2+P_k]^{1/2}\} r dr$$

$$+ \delta_{m\nu} \, n_2(-P_k) \frac{\partial^2}{\partial x_\nu \partial x_1} \int_0^\infty \frac{J_0(\Omega r)}{[r^2+P_k]^{1/2}} \exp\{-|x_1|[r^2+P_k]^{1/2}\} r dr \Bigg\},$$

$$m = 1,2,3, \quad \nu = 2,3.$$

The integrals in (4.15) are known and may be found in [10], p. 31, no. 22. They are reproduced here:

$$\int_0^\infty \frac{r\, J_0(\Omega r)}{[r^2+P_k]^{1/2}} \exp\{-|x_1|[r^2 + P_k]^{1/2}\} dr = \frac{1}{R} \exp\{-R(P_k)^{1/2}\},$$

$$\int_0^\infty \frac{r^3 J_0(\Omega r)}{[r^2+P_k]^{1/2}} \exp\{-|x_1|[r^2+P_k]^{1/2}\}\, dr = \tag{4.16}$$

$$\frac{1}{\Omega} \frac{\partial}{\partial \Omega} \Big\{ \frac{\Omega^2}{R^2} (P_k)^{1/2} \Big[1 + \frac{1}{R(P_k)^{1/2}} \Big] \exp\{-R(P_k)^{1/2}\} \Big\},$$

provided

$$\mathrm{Re}(P_k)^{1/2} > 0, \quad k = 1,2,3,4$$

where

$$R = \sqrt{x_1^2 + \Omega^2}.$$

Incorporating (4.16) into (4.15) we have

$$\hat{w}_m(x_1,x_2,x_3,p) = \frac{\bar{\rho}_1 \delta_{m1}}{4\pi R} \sum_{k=1}^{4} \frac{n_1(-P_k) \exp\{-R(P_k)^{1/2}\}}{\beta_3\mu[K_1(K_2 + \frac{\theta_2}{p}) - \frac{\theta_1^2}{p}] \prod_{\substack{s=1 \\ s\neq k}}^{4} (P_k - P_s)} +$$

$$- \frac{\bar{\rho}_1 \delta_{m1}}{4\pi\Omega} \frac{\partial}{\partial\Omega} \left\{ \frac{\Omega^2}{R^2} \sum_{k=1}^{4} \frac{n_2(-P_k)\exp\{-R(P_k)^{1/2}\}}{\beta_3\mu[K_1(K_2 + \frac{\Theta_2}{p}) - \frac{\Theta_1^2}{p}] \prod_{\substack{s=1 \\ s\neq k}}^{4} (P_k - P_s)} \left[(P_k)^{1/2} + \frac{1}{R} \right] \right\}$$

$$- \frac{\bar{\rho}_1 \delta_{m\nu}}{4\pi} \frac{\partial^2}{\partial x_\nu \partial x_1} \left\{ \frac{1}{R} \sum_{k=1}^{4} \frac{n_2(-P_k)\exp\{-R(P_k)^{1/2}\}}{\beta_3\mu[K_1(K_2 + \frac{\Theta_2}{p}) - \frac{\Theta_1^2}{p}] \prod_{\substack{s=1 \\ s\neq k}}^{4} (P_k - P_s)} \right\},$$

$$m = 1,2,3, \quad \nu = 2,3, \quad \mathrm{Re}(P_k)^{1/2} > 0, \quad k = 1,2,3,4. \tag{4.17}$$

Equation (4.17) is the final expression for the completed Fourier inversion. Because of the similarity between the expressions (3.9) and (3.10), the fluid velocity components may be inverted in exactly the same manner with only notational differences. If in (4.17) we replace n_1 and n_2 by m_1, m_2 defined by,

$$\begin{aligned} m_1(-P_k) = {} & \Delta_2(-P_k)\ \alpha\ p[\Delta_3(-P_k) + (K_1 - \beta_3)P_k] \\ & - \Delta_5(-P_k)\ \alpha\ [\Theta_1 P_k - \Delta_5(-P_k)] \\ & - P_k\ \Delta_4(-P_k)[\Theta_1\Delta_1(-P_k) - \alpha\ p(K_2 - \mu + \frac{\Theta_2}{p})], \end{aligned} \tag{4.18a}$$

$$\begin{aligned} m_2(-P_k) = {} & (K_1 - \beta_3)\ \alpha\ p\ \Delta_2(-P_k) - \alpha\ \Theta_1\ \Delta_5(-P_k) \\ & - \Delta_4(-P_k)[\Theta_1\Delta_1(-P_k) - \alpha\ p(K_2 - \mu + \frac{\Theta_2}{p})] \end{aligned} \tag{4.18b}$$

then it is readily verified that $\hat{v}_m(x_1,x_2,x_3,p)$ is given by

$$\hat{v}_m(x_1,x_2,x_3,p) = \frac{-\bar{\rho}_1\delta_{m1}}{4\pi R} \sum_{k=1}^{4} \frac{m_1(-P_k)\exp\{-R(P_k)^{1/2}\}}{\beta_3\mu[K_1(K_2 + \frac{\Theta_2}{p}) - \frac{\Theta_1^2}{p}] \prod_{\substack{s=1 \\ s\neq k}}^{4} (P_k - P_s)} +$$

$$- \frac{\bar{\rho}_1 \delta_{m1}}{4\pi\Omega} \frac{\partial}{\partial\Omega} \left\{ \frac{\Omega^2}{R^2} \sum_{k=1}^{4} \frac{m_2(-P_k)\exp\{-R(P_k)^{1/2}\}}{\beta_3\mu[K_1(K_2 + \frac{\Theta_2}{p}) - \frac{\Theta_1^2}{p}] \prod_{\substack{s=1 \\ s \neq k}}^{4} (P_k - P_s)} [(P_k)^{1/2} + \frac{1}{R}] \right\}$$

$$+ \frac{\bar{\rho}_1 \delta_{m\nu}}{4\pi} \frac{\partial^2}{\partial x_\nu \partial x_1} \left\{ \frac{1}{R} \sum_{k=1}^{4} \frac{m_2(-P_k)\exp\{-R(P_k)^{1/2}\}}{\beta_3\mu[K_1(K_2 + \frac{\Theta_2}{p}) - \frac{\Theta_1^2}{p}] \prod_{\substack{s=1 \\ s \neq k}}^{4} (P_k - P_s)} \right\},$$

$$m = 1,2,3, \quad \nu = 2,3, \quad \mathrm{Re}(P_k)^{1/2} > 0, \quad k = 1,2,3,4. \tag{4.19}$$

5. <u>Laplace inversion</u>: <u>Exact solution when $\alpha = 0$</u>. To proceed with the Laplace inversion our first task is to make the multi-valued functions R_α and $(P_k)^{1/2}$, defined in (4.2), (4.4) and (4.6), single-valued in the p-plane. An examination of (4.4) and (4.6) reveals that this is a formidable task due partly to the order of the polynomials in R_1, R_2 and due partly to the material-constant coefficients in these expressions. Although it is clearly possible to obtain the exact branch points of R_1 and R_2 and also of the $(P_k)^{1/2}$, we choose instead to consider the case in which the diffusive resistance parameter α is negligible, i.e. $\alpha = 0$. With this approximation, inversion of (4.17) and (4.19) can be accomplished much more readily. Information concerning the case when $\alpha \neq 0$ can be obtained by, say, a perturbation method-regular or singular-using the solution found here. This is not attempted here.

By setting $\alpha = 0$ in (4.3) to (4.6) we see that (4.2) becomes

$$P_1 = \frac{\bar{\rho}_2 p}{\mu} \quad , \quad P_2 = \frac{\bar{\rho}_1 p^2}{\beta_3} \; , \tag{5.1}$$

$$P_{3,4} = \frac{p^2[\bar{\rho}_2 K_1 + \bar{\rho}_1(K_2 p + \theta_2)] \pm R_2(p)}{2[K_1(K_2 p + \theta_2) - \theta_1^2]} \tag{5.2}$$

$$R_2(p) = p^2[\{K_1\bar{\rho}_2 - (K_2 p + \theta_2)\bar{\rho}_1\}^2 + 4\,\bar{\rho}_1\bar{\rho}_2\theta_1^2]^{1/2}. \tag{5.3}$$

Also, by (3.13), (4.9), and (4.10),

$$n_1(-P_k) = -\frac{\beta_3\mu}{p}\,[P_k - P_1][P_k - P_2][(K_2 p + \theta_2)P_k - \bar{\rho}_2 p^2], \tag{5.4}$$

$$n_2(-P_k) = \frac{\mu}{p}[P_k - P_1][\{(K_1-\beta_3)(K_2 p + \theta_2) - \theta_1^2\}P_k - (K_1-\beta_3)p^2\bar{\rho}_2], \tag{5.5}$$

$$n_1(-P_1) = n_1(-P_2) = n_2(-P_1) = 0, \; k = 1,2,3,4. \tag{5.6}$$

To render $(P_3)^{1/2}$ and $(P_4)^{1/2}$ defined in (5.2) singlevalued we must first examine $R_2(p)$ of (5.3). The branch points of R_2 occur at

$$K_2 p = \frac{\bar{\rho}_2 K_1 - \bar{\rho}_1\theta_2}{\bar{\rho}_1} \pm 2i\theta_1\sqrt{\frac{\bar{\rho}_2}{\bar{\rho}_1}} \; . \tag{5.7}$$

To position these points in the p-plane we shall assume (recall (2.6), (2.13) and (2.14))

$$\frac{K_1}{\bar{\rho}_1} > \frac{\theta_2}{\bar{\rho}_2} \geq 0 \; , \; \theta_1 \geq 0, \quad K_2 \geq 0, \tag{5.8}$$

$$K_1\theta_2 > \theta_1^2,$$

and define

$$c_1^2 = \frac{K_1}{\bar{\rho}_1} \; , \; c_2^2 = \frac{\theta_2}{\bar{\rho}_2} \; , \; c_3 = \frac{\theta_1}{\sqrt{\bar{\rho}_1\bar{\rho}_2}} \; , \; c_4^2 = \frac{K_2}{\bar{\rho}_2} \tag{5.9}$$

$$v_s^2 = \frac{\beta_3}{\bar{\rho}_1} \; .$$

With these definitions $R_2(p)$ has branch points at

$$c_4^2 p = c_1^2 - c_2^2 \pm 2ic_3. \tag{5.10}$$

We now cut the p-plane along the straight line

$$p = \frac{c_1^2 - c_2^2}{c_4^2} \pm \frac{2ic_3\omega}{c_4^2}, \quad -1 \le \omega \le 1 \tag{5.11}$$

and define that branch of (5.3) such that

$$\{[c_4^2 p - (c_1^2 - c_2^2)]^2 + 4c_3^2\}^{1/2} = +\sqrt{[c_4^2 p - (c_1^2-c_2^2)]^2 + 4c_3^2} \tag{5.12}$$

when p is large and real.

Introduce (5.9) into (5.2) and define

$$\begin{aligned} F_1(p) &= c_4^2 p + c_1^2 + c_2^2 + \{[c_4^2 p - (c_1^2 - c_2^2)]^2 + 4c_3^2\}^{1/2} \\ F_2(p) &= c_4^2 p + c_1^2 + c_2^2 - \{[c_4^2 p - (c_1^2 - c_2^2)]^2 + 4c_3^2\}^{1/2} \end{aligned} \tag{5.13}$$

$$v_1^2 = c_2^2 - v_s^2 - \frac{c_3^2}{c_1^2 - v_s^2}, \quad v_2^2 = c_2^2 - \frac{c_3^2}{c_1^2}.$$

Then (5.2) may be written as

$$P_3(p) = \frac{p^2 F_1(p)}{2c_1^2(c_4^2 p + v_2^2)}, \quad P_4 = \frac{p^2 F_2(p)}{2c_1^2(c_4^2 p + v_2^2)} \tag{5.14}$$

or as

$$P_3(p) = \frac{2p^2}{F_2(p)}, \quad P_4 = \frac{2p^2}{F_1(p)} \tag{5.15}$$

if we use

$$F_1(p)F_2(p) = 4c_1^2(c_4^2 p + v_2^2). \tag{5.16}$$

In addition to the branch points (5.10) other possible branch points of $(P_3)^{1/2}$ and $(P_4)^{1/2}$ occur at the zeros of F_1 or F_2. The point which satisfies this requirement is

$$p = -\frac{v_2^2}{c_4^2} \tag{5.17}$$

which by (5.8) is real and negative.

Along the negative real axis of the p-plane the radical in (5.13) is given by

$$\{\quad\}^{1/2} = -\sqrt{(c_4^2 p + c_1^2 - c_2^2)^2 + 4c_3^2} \tag{5.18}$$

and at $p = -v_2^2/c_4^2$ this becomes

$$\{\quad\}^{1/2} = -\ (c_1^2 + c_3^2/c_1^2)\,. \tag{5.19}$$

By (5.17) and (5.19) , (5.13) at $-v_2^2/c_4^2$ gives

$$F_1(p) = 0; \quad F_2(p) = 2(c_1^2 + c_3^2/c_1^2) > 0. \tag{5.20}$$

Therefore to render $[F_1(p)]^{1/2}$ single-valued we cut along the line $-\infty < -\ \mathrm{Re}(p) \le -v_2^2/c_4^2$ and define that branch such that

$$F_1^{1/2}(p) = +\sqrt{F_1(p)} \quad \text{when } p \text{ is real and positive.} \tag{5.21)$'$}$$

On the other hand no cut other than (5.11) is necessary to render $F_2^{1/2}(p)$ single-valued and so when p is real, $p > (c_1^2 - c_2^2)|c_4^2$,

$$\begin{aligned} F_1^{1/2}(p) &= +\sqrt{c_4^2 p + c_1^2 + c_2^2 + \sqrt{[c_4^2 p - (c_1^2 - c_2^2)]^2 + 4c_3^2}} \\ F_2^{1/2}(p) &= +\sqrt{c_4^2 p + c_1^2 + c_2^2 - \sqrt{[c_4^2 p - (c_1^2 - c_2^2)]^2 + 4c_3^2}}\,. \end{aligned} \tag{5.21}$$

With all of these definitions in mind we are now in a position to consider the Laplace inversion of (4.17). To do so we rewrite (4.17) in the notation just developed. Therefore, by (5.1) to (5.6), (5.8), (5.9), (5.13) to (5.15) we have

$$\hat{w}_m(x_1,x_2,x_3,p) = \frac{\delta_{m1}(C_1^2 - v_s^2)}{4\pi\Omega} \frac{\partial}{\partial\Omega}\left[\frac{\Omega^2}{R^2}\{\bar{I}_{41}(R,p) - \bar{I}_{31}(R,p)\}\right]$$

$$- \frac{\delta_{m1}}{4\pi R}\{\bar{I}_{42}(R,p) - \bar{I}_{32}(R,p)\}$$

$$+ \frac{\delta_{m\nu}(C_1^2 - v_s^2)}{4\pi} \frac{\partial^2}{\partial x_\nu \partial x_1}\left[\frac{\bar{I}_{43}(R,p) - \bar{I}_{33}(R,p)}{R}\right]$$

$$- \bar{I}_{m\nu}(R,p), \quad m = 1,2,3, \quad \nu = 2,3. \tag{5.22}$$

The terms in (5.22) are defined as follows:

$$\bar{I}_{m\nu}(R,p) = \frac{\delta_{m1}}{4\pi\Omega}\frac{\partial}{\partial\Omega}\left[\frac{\Omega^2}{R^2}\left\{\frac{1}{v_s p} + \frac{1}{p^2 R}\right\}\exp\left(\frac{Rp}{v_s}\right)\right]$$
$$+ \frac{\delta_{m\nu}}{4\pi}\frac{\partial^2}{\partial x_\nu \partial x_1}\left[\frac{1}{p^2 R}\exp\left(-\frac{pR}{v_s}\right)\right]; \tag{5.23}$$

$$\bar{I}_{41}(R,p) = \frac{2(C_4^2 p + v_1^2 + v_s^2) - F_1(p)}{p^2[2v_s^2 - F_1(p)]\{[C_4^2 p - (C_1^2 - C_2^2)]^2 + 4C_3^2\}^{1/2}}\left[\frac{pF_2^{1/2}(p)}{\sqrt{2}\,C_1(C_4^2 p + v_2^2)^{1/2}} + \right.$$
$$\left. + \frac{1}{R}\right]\exp\left\{-\frac{\sqrt{2}pR}{F_1^{1/2}(p)}\right\}; \tag{5.24}$$

$$\bar{I}_{42}(R,p) = \frac{2(C_4^2 p + C_2^2) - F_1(p)}{F_1(p)\{[C_4^2 p - (C_1^2 - C_2^2)]^2 + 4C_3^2\}^{1/2}}\exp\left\{-\frac{\sqrt{2}pR}{F_1^{1/2}(p)}\right\}. \tag{5.25}$$

From $\bar{I}_{41}$ we obtain $\bar{I}_{31}(R,p)$ by interchanging F_1 and F_2 wherever they appear. Similarly, to obtain $\bar{I}_{32}(R,p)$ replace $F_1(p)$ with $F_2(p)$ in $\bar{I}_{42}(R,p)$. Finally, $\bar{I}_{43}(R,p)$ is the same as $\bar{I}_{41}(R,p)$ with the term

$$\frac{p\,F_2^{1/2}(p)}{\sqrt{2}\,C_1(C_4^2 p + v_2^2)^{1/2}} + \frac{1}{R}$$

replaced by 1 and $\bar{I}_{33}(R,p)$ is then found by replacing F_1 by F_2 in $\bar{I}_{43}(R,p)$.

This section is closed by recalling the inversion integral (3.2b) and by replacing the Bromwich contour by means of an appropriate path in the cut p-plane as in figure 1. Accordingly we define the inverse of $\bar{I}(R,p)$ to be

$$I(R,t) = \lim_{\omega\to\infty} \frac{1}{2\pi i} \int_{x-i\omega}^{x+i\omega} \bar{I}(R,p)e^{pt}dp \tag{5.26}$$

in which $\bar{I}(R,p)$ refers to any of the terms in (5.22) and the path is chosen such that all singularities of $\bar{I}(R,p)e^{pt}$ lie to the left of x, i.e.

$$x > \frac{c_1^2 - c_2^2}{c_4^2} . \tag{5.27}$$

In the integrals (5.26) of terms in (5.22) we must know the order of the integrand for p real and large. Then, in view of (5.12), (5.21) and (5.22) to (5.25) we have for p real and large

$$\left.\begin{aligned}
(\bar{I}_{41}(R,p)e^{pt},\ \bar{I}_{43}(R,p)e^{pt}) &\sim e^{\frac{pt-Rp^{1/2}}{c_4}}\left[\frac{1}{p^3} + O(p^{-4})\right],\\
(\bar{I}_{31}(R,p)e^{pt},\ \bar{I}_{33}(R,p)e^{pt}) &\sim e^{p(t-\frac{R}{c_1})}\left[\frac{1}{p^3} + O(p^{-4})\right],\\
\bar{I}_{42}(R,p)e^{pt} &\sim e^{\frac{pt-Rp^{1/2}}{c_4}}\left[\frac{1}{p} + O(\frac{1}{p^2})\right]\\
\bar{I}_{32}(R,p)e^{pt} &\sim e^{p(t-\frac{R}{c_1})}\left[\frac{1}{p} + O(\frac{1}{p^2})\right]
\end{aligned}\right\} \tag{5.28}$$

6. <u>Laplace inversion continued</u>. We begin the inversion of (5.22) by considering (5.24) first. The process is given in detail for $\bar{I}_{41}(R,p)$ and then the inverted I_{42} and I_{43}

expressions are inferred from these results. A similar procedure is used on $\bar{I}_{31}$, $\bar{I}_{32}$ and $\bar{I}_{33}$.

By (5.24), (5.26) and (5.28) we have for $t > 0$, (refer to figure 1)

$$\frac{1}{2\pi i}\oint_C \bar{I}_{41}(R,p)e^{pt}dp = I_{41}(R,t)$$

$$+\frac{1}{2\pi i}\left\{\int_{BD}+\int_{DE}+\int_{EW}+\int_{WX}+\int_{YT}+\int_{TM}+\int_{MN}\right.$$

$$\left.+\int_{NPQ}+\int_{QR}+\int_{RS}+\int_{SY'}+\int_{X'W'}+\int_{W'F}+\int_{FG}+\int_{GA}\right\}\bar{I}_{41}(R,p)e^{pt}dp$$

$$= \text{Res}\left\{\bar{I}_{41}(R,p)e^{pt} \text{ at } p = 0\right\} \tag{6.1}$$

to be taken in the limit as $\omega \to \infty$, $R_3 \to \infty$ and the circular arcs EW, W′F, MN and QR shrink to zero.

By the fact that $F_1, F_2, F_1^{1/2}, F_2^{1/2}$ are single-valued inside and on C and by Jordan's lemma we conclude that the integrals of (6.1) over the arcs WX, X′W′, YT, SY′, BD and GA vanish.

On the arcs DE and FG, by (5.13) to (5.18), we set

$$\left.\begin{aligned}
&c_4^2p + v_2^2 = re^{i\pi}, \; [c_4^2p + v_2^2]^{1/2} = ir^{1/2}, \; \epsilon_0 \le r < \infty \text{ on DE,}\\
&c_4^2p + v_2^2 = re^{-i\pi}, \; [c_4^2p + v_2^2]^{1/2} = -ir^{1/2}, \; \epsilon_0 \le r < \infty \text{ on FG,}\\
&\{[c_4^2p + c_2^2 - c_1^2]^2 + 4c_3^2\}^{1/2} = -\sqrt{\left(r + c_1^2 - \frac{c_3^2}{c_1^2}\right)^2 + 4c_3^2} \equiv\\
&\quad - r_1(r) \text{ on DE and FG,}
\end{aligned}\right\} \tag{6.2*}$$

---

*Here ϵ_0 is the radius of the circle $|c_4^2p + v_2^2| = \epsilon_0$, which in the limit will approach zero.

$$\left.\begin{aligned} F_1(p) &= c_1^2 + c_3^2/c_1^2 - r - r_1(r) \le 0, \ \epsilon_0 \le r < \infty \text{ on DE and FG}, \\ F_2(p) &= c_1^2 + c_3^2/c_1^2 - r + r_1(r) \ge 0, \ \epsilon_0 \le r < \infty \text{ on DE and FG}, \\ F_2^{\ 1/2}(p) &= +\sqrt{F_2(p)} \equiv \Gamma_1^{\ 1/2}(r) \quad \text{on DE and FG.} \end{aligned}\right\} \quad (6.2)$$

Then, in (6.1),

$$\frac{1}{2\pi i}\left[\int_{DE} + \int_{FG}\right] \bar{I}_{41}(R,p)e^{pt}dp = J_1(R,t) =$$

$$\frac{c_4^2}{\pi}\int_{\epsilon_0}^{\infty} \frac{2(v_1^2 + v_s^2 - v_2^2 - r) - F_1(r)}{(r + v_2^2)^2[F_1(r) - 2v_s^2]r_1(r)} \left\{ \frac{(r+v_2^2)\Gamma_1^{\ 1/2}(r)}{\sqrt{2}\ c_4^2\ c_1 r^{1/2}} \right.$$

$$\cdot\cos\left[\frac{(r+v_2^2)\Gamma_1^{\ 1/2}(r)R}{\sqrt{2}\ c_1 c_4^2 r^{1/2}}\right] - \frac{1}{R}\sin\left[\left.\frac{(r+v_2^2)\Gamma_1^{\ 1/2}(r)R}{\sqrt{2}\ c_1 c_4^2 r^{1/2}}\right]\right\}$$

$$\cdot\exp\left[\frac{-t(r+v_2^2)}{c_4^2}\right] dr. \tag{6.3}$$

According to figure 1, we next consider the arcs EW and W′F which are parts of the circle

$$c_4^2 p + v_2^2 = \epsilon_0 e^{i\phi} \qquad - \quad \pi < \phi \le \pi \tag{6.4}$$

taken in the clockwise sense. If one uses (6.4) directly in (6.1), and uses the branches of $F_1^{\ 1/2}$, $F_2^{\ 1/2}$ defined by (5.21), the contribution from the resulting integral over EW W′F is not readily established. Instead, we consider the transformation

$$c_4^2 p + v_2^2 = u^2 \tag{6.5}$$

which, as shown in figure 2, maps (6.4) onto the semi-circle C' with the direction still clockwise. Using (6.5) in (6.1) we define

$$I_{\underset{\sim}{R}}(R,t) = \frac{1}{2\pi i}\oint_{C'} \bar{I}_{41}(R,u)\,e^{\frac{t[u^2-v_2^2]}{c_4^2}}\,du. \tag{6.6}$$

Again consider (6.1) but now change the branch of $F_1^{1/2}(p)$ to be $-\sqrt{F_1(p)}$ when p is large and positive. If in (6.4) we were to go over the circular arc twice, i.e. use

$$c_4^2 p + v_2^2 = \epsilon_0 e^{i(\phi+2\pi)} \qquad -\pi < \phi \le \pi \tag{6.7}$$

then (6.7) under the mapping (6.5) is carried into the semi-circle C'' shown in figure 2, and as we did in (6.6) we define

$$I_{\underset{\sim}{L}}(R,t) = \frac{1}{2\pi i}\oint_{C''} \bar{I}_{41}(R,u)\,e^{\frac{t[u^2-v_2^2]}{c_4^2}}\,du. \tag{6.8}$$

If we reverse the directions in (6.6) and (6.8) and add, then by the residue theorem

$$I_{\underset{\sim}{L}} + I_{\underset{\sim}{R}} = -\,\mathrm{Res}\left\{ \bar{I}_{41}(R,u)\,e^{\frac{t(u^2-v_2^2)}{c_4^2}} \quad \text{at} \quad u = 0 \right\} \tag{6.9}$$

where $u = 0$ is the only singularity enclosed by the contours $C' + C''$.

Consider now $I_{\underset{\sim}{L}}(R,t)$ as given in (6.8). This integral is precisely (6.1) taken over the arc $EWW'F$ in the p-plane if we were to use the second branch of $F_1^{1/2}(p)$. As such it is little trouble to establish the fact that as $\epsilon_0 \to 0$ so does $I_{\underset{\sim}{L}}(R,t) \to 0$. Hence

$$\lim_{\epsilon_0 \to 0} \frac{1}{2\pi i} \oint_{EWW'F} \bar{I}_{41}(R,p)e^{pt}dp =$$

$$- \text{Res}\left\{\bar{I}_{41}(R,u)e^{\frac{t(u^2-v_2^2)}{C_4^2}} \quad \text{at} \quad u = 0\right\}$$

where u is given by (6.5). Explicitly, by (5.18), (5.21) and (5.24)

$$\lim_{\epsilon_0 \to 0} \frac{1}{2\pi i} \oint_{EWW'F} \bar{I}_{41}(R,p)e^{pt}\,dp =$$

$$- 2\,C_4^2 \,\text{Res}\left[\frac{2(v_1^2 + v_s^2 + u^2 - v_2^2) - F_1(u)}{(u^2 - v_2^2)^2[F_1(u) - v_s^2]r_1(u)} \left\{ \frac{(u^2 - v_2^2)F_2^{1/2}(u)}{\sqrt{2}\,C_1\,C_4^2}\right.\right.$$

$$\left.\left. + \frac{u}{R}\right\} \exp\left\{\frac{u^2-v_2^2}{C_4^2}\left(t - \frac{R\,F_2^{1/2}(u)}{\sqrt{2}\,C_1\,u}\right)\right\} \text{at} \quad u = 0\right] \qquad (6.10)$$

where

$$\left.\begin{aligned} r_1(u) &= \sqrt{(u^2-C_1^2+C_3^2/C_1^2)^2 + 4C_3^2}\,, \\ F_1(u) &= C_1^2+C_3^2/C_1^2 + u^2 - r_1(u)\,, \\ F_2(u) &= C_1^2+C_3^2/C_1^2 + u^2 + r_1(u)\,. \end{aligned}\right\} \qquad (6.11)$$

Turn now to the integral to be taken around the cut (5.11). We observe that it can be shown that (6.1) over the arcs MN and QR constitute nothing as the radii of these arcs vanish.

Define

$$\left.\begin{aligned} C_4^2p - (C_1^2 - C_2^2) &= 2iC_3\omega,\ 0 \le \omega < 1, \text{ on TM and NP}, \\ C_4^2p - (C_1^2 - C_2^2) &= 2iC_3\omega.\ 0 \le \omega < 1, \text{ on RS and PQ}, \end{aligned}\right\} \qquad (6.12)$$

$$\left.\begin{aligned}&\left\{[C_4^2 p - (C_1^2-C_2^2)]^2 + 4C_3^2\right\}^{1/2} = -2C_3\sqrt{1-\omega^2} \quad \text{on TM and RS,}\\&\left\{[C_4^2 p - (C_1^2-C_2^2)]^2 + 4C_3^2\right\}^{1/2} = 2C_3\sqrt{1-\omega^2} \quad \text{on NP and PQ,}\end{aligned}\right\}\quad(6.12)$$

$$\left.\begin{aligned}&F_1(p) = 2\Phi_1(\omega)e^{i\Theta_1},\ F_2(p) = 2\Phi_2(\omega)e^{i\Theta_2} \quad \text{on TM,}\\&F_1(p) = 2\Phi_1(\omega)e^{-i\Theta_1},\ F_2(p) = 2\Phi_2(\omega)e^{-i\Theta_2} \quad \text{on RS,}\\&F_1(p) = 2\Phi_2(\omega)e^{i\Theta_2},\ F_2(p) = 2\Phi_1(\omega)e^{i\Theta_1} \quad \text{on NP,}\\&F_1(p) = 2\Phi_2(\omega)e^{-i\Theta_2},\ F_2(p) = 2\Phi_1(\omega)e^{-i\Theta_1} \quad \text{on PQ}\end{aligned}\right\}\quad(6.13)$$

where

$$\begin{aligned}\Phi_1(\omega) &= \sqrt{[C_1^2 - C_3\sqrt{1-\omega^2}\,]^2 + C_3^2\omega^2},\\\Phi_2(\omega) &= \sqrt{[C_1^2 + C_3\sqrt{1-\omega^2}\,]^2 + C_3^2\omega^2},\end{aligned}\qquad(6.14)$$

$$\tan\Theta_1 = \frac{C_3\omega}{C_1^2-C_3\sqrt{1-\omega^2}}, \quad \tan\Theta_2 = \frac{C_3\omega}{C_1^2+C_3\sqrt{1-\omega^2}}, \quad 0 \le \Theta_1,\Theta_2 < \frac{\pi}{2}.$$

After some algebra, by (6.1) and (6.12) to (6.14), we have when $t \ge 0$,

$$\frac{1}{2\pi i}\left[\int_{TM} + \int_{RS}\right] \bar{I}_{41}(R,p)e^{pt}\,dp = J_2(R,t) =$$

$$\frac{C_4^2}{\pi}\,\underline{\mathrm{Re}}\int_0^1 \frac{v_1^2+v_s^2+C_1^2-C_2^2+2iC_3\omega - \Phi_1(\omega)e^{i\Theta_1}}{\sqrt{1-\omega^2}\,[C_1^2-C_2^2+2iC_3\omega]^2[\Phi_1 e^{i\Theta_1}-v_s^2]}\left\{\frac{C_1^2-C_2^2+2iC_3\omega}{C_4^2\,\Phi_1^{1/2}(\omega)e^{\frac{i\Theta_1}{2}}} + \frac{1}{R}\right\}\exp\left[\frac{C_1^2-C_2^2+2iC_3\omega}{C_4^2}\left(t - \frac{Re^{\frac{-i\Theta_1}{2}}}{\Phi_1^{1/2}(\omega)}\right)\right]d\omega, \qquad (6.15)$$

and

$$\frac{1}{2\pi i}\left[\int_{NP} + \int_{PQ}\right] \bar{I}_{41}(R,p)e^{pt}\,dp = J_3(R,t) =$$

$$\frac{c_4^2}{\pi}\,\underline{\mathrm{Re}}\int_0^1 \frac{v_1^2+v_s^2+c_1^2-c_2^2+2ic_3\omega-\Phi_2(\omega)e^{i\theta_2}}{\sqrt{1-\omega^2}\,[c_1^2-c_2^2+2ic_3\omega]^2[\Phi_2e^{i\theta_2}-v_s^2]}\left\{\frac{c_1^2-c_2^2+2ic_3\omega}{c_4^2\Phi_2^{1/2}(\omega)e^{i\theta_2/2}}\right.$$

$$\left.+\frac{1}{R}\right\}\exp\left[\frac{c_1^2-c_2^2+2ic_3\omega}{c_4^2}\left(t-\frac{Re^{-i\theta_2/2}}{\Phi_2^{1/2}(\omega)}\right)\right]d\omega. \tag{6.16}$$

In (6.15) and (6.16) the expression "$\underline{\mathrm{Re}}\int$" means "the real part of $\int$."

Returning to (6.1), and in view of (6.3), (6.10), (6.15) and (6.16), we have that as $R_3 \to \infty$, $\epsilon_0 \to 0$ and $\omega \to \infty$, then for $t > 0$,

$$I_{41}(R,t) + J_1(R,t) + J_2(R,t) + J_3(R,t) =$$

$$\mathrm{Res}\left\{\bar{I}_{41}(R,p)e^{pt} \text{ at } p=0\right\} + \mathrm{Res}\left\{\bar{I}_{41}(R,u)e^{\frac{t(u^2-v_2^2)}{c_4^2}} \text{ at } u=0 \quad \text{where } c_4^2p+v_2^2=u^2\right\}. \tag{6.17}$$

A word about the residues in (6.17). The first term, i.e. the residue at $p = 0$, can be computed immediately since an examination of (5.24) reveals that one term in (5.24) has a simple pole at $p = 0$ while the other term in (5.24) has a double pole. Thus, bearing in mind (5.18) and (5.13)

Res $\{\bar{I}_{41}(R,p)e^{pt}$ at $p = 0\} =$

$$
\begin{aligned}
&-\frac{t}{R}\ \frac{[2(v_1^2 + v_s^2) - F_1(0)]}{[2v_s^2 - F_1(0)]\sqrt{(c_1^2-c_2^2)^2 + 4c_3^2}}\ H(t) \\
&- \frac{[\{c_1^2 - c_2^2\}^2 + 4c_3^2][2c_4^2\{2v_s^2 - F_1(0)\} + 2v_1^2\ F_1'(0)]}{R[2v_s^2 - F_1(0)]^2\sqrt{(c_1^2 - c_2^2)^2 + 4c_3^2}}\ H(t) \\
&- \frac{c_4^2(c_1^2 - c_2^2)(2v_s^2 - F_1(0))(2(v_1^2 + v_s^2) - F_1(0))}{R[2v_s^2 - F_1(0)]^2\ [(c_1^2 - c_2^2)^2 + 4c_3^2]^{3/2}}\ H(t). \qquad (6.18)
\end{aligned}
$$

Here $F_1(0) = d\ F_1(p)/d\ p$ at $p = 0$.

On the other hand, the second residue in (6.17), i.e. (6.10) must be computed by multiplying the Laurent series expansions together and collecting those terms involving u^{-1}. The author has not succeeded in summing the resulting series at this time. For a particular material the series can be summed numerically on a computer.

To continue the Laplace inversion of $\hat{w}_m$ given in (5.22) we next consider the term $\bar{I}_{43}(R,p)$ which is the same as $\bar{I}_{41}(R,p)$ of (5.24) with the term

$$
\frac{p\ F_2{}^{1/2}(p)}{\sqrt{2}\ c_1(c_4^2 p + v_2^2)^{1/2}} + \frac{1}{R}
$$

replaced by 1. In view of (5.28), (6.1) and the work just completed on $\bar{I}_{41}(R,p)$, we can write down the final value of $I_{43}(R,t)$ for $t > 0$. Thus, by (6.1), (6.3), (6.10), (6.15), and (6.18) we find in the limit as $R_3 \to \infty$, $\epsilon \to 0$, $w \to \infty$

$$I_{43}(R,t) + J_4(R,t) + J_5(R,t) + J_6(R,t) = \mathrm{Res}\left\{\bar{I}_{43}(R,p)e^{pt} \text{at } p=0\right\} + \mathrm{Res}\left\{\bar{I}_{43}(R,u)\,e^{\frac{t(u^2-v_2^2)}{c_4^2}} \text{ at } u = 0 \right.$$

$$\left. \text{where } c_4^2 p + v_2^2 = u^2\right\} \tag{6.19}$$

where

$$J_4(R,t) = -\frac{c_4^2}{\pi}\int_0^\infty \frac{2(v_1^2+v_s^2-v_2^2-r)-F_1(r)}{(r+v_2^2)^2[F_1(r)-2v_s^2]r_1(r)} \sin\left[\frac{(r+v_2^2)\Gamma_1^{1/2}(r)R}{\sqrt{2}c_1c_4^2r^{1/2}}\right]$$

$$\cdot \exp\left[\frac{-t(r+v_2^2)}{c_4^2}\right] dr, \tag{6.20}$$

with $F_1(r)$, $r_1(r)$, and $\Gamma_1^{1/2}(r)$ given by (6.2), and,

$$J_5(R,t) = \frac{c_4^2}{\pi}\,\underline{\mathrm{Re}}\int_0^1 \frac{v_1^2+v_s^2+c_1^2-c_2^2+2ic_3\omega-\Phi_1(\omega)e^{i\theta_1}}{\sqrt{1-\omega^2}\,[c_1^2-c_2^2+2ic_3\omega]^2[\Phi_1e^{i\theta_1}-v_s^2]}$$

$$\cdot \exp\left\{\frac{c_1^2-c_2^2+2ic_3\omega}{c_4^2}\left(t - \frac{Re^{-i\theta_1/2}}{\Phi_1^{1/2}(\omega)}\right)\right\} d\omega \tag{6.21}$$

$$J_6(R,t) = \frac{c_4^2}{\pi}\,\underline{\mathrm{Re}}\int_0^1 \frac{v_1^2+v_s^2+c_1^2-c_2^2+2ic_3\omega-\Phi_2(\omega)e^{i\theta_2}}{\sqrt{1-\omega^2}[c_1^2-c_2^2+2ic_3\omega]^2[\Phi_2e^{i\theta_2}-v_s^2]}$$

$$\cdot \exp\left\{\frac{c_1^2-c_2^2+2ic_3\omega}{c_4^2}\left(t - \frac{Re^{-i\theta_2/2}}{\Phi_2^{1/2}(\omega)}\right)\right\} d\omega, \tag{6.22}$$

where Φ_1, Φ_2, Θ_1 and Θ_2 are given in (6.12) to (6.14).

The residue term in (6.19) at $p = 0$ is given by

$$\text{Res } \{\bar{I}_{43}(R,p)e^{pt} \text{ at } p = 0\} =$$

$$- \left(t - \frac{\sqrt{2}\, R}{F_1^{1/2}(0)}\right) \left[\frac{2(v_1^2 + v_s^2) - F_1(0)}{[2v_s^2 - F_1(0)]r_1(0)} \right] H(t)$$

$$- \frac{1}{[2v_s^2 - F_1(0)]^2 r_1{}^3(0)} \Big[C_4^2\, r_1(0)(2v_s^2 - F_1(0))\ (F_2(0) - 2C_1^2)$$

$$+ (2(v_1^2 + v_s^2) - F_1(0))\ \{r_1^2(0)\ F_1(0) + C_4^2(C_1^2 - C_2^2)(2v_s^2 - F_1(0))\}\Big] H(t), \tag{6.23}$$

where $F_1(0) = C_1^2 + C_2^2 - r_1(0)$, $F_2(0) = C_1^2 + C_2^2 + r_1(0)$

and $r_1(0) = \sqrt{(C_1^2 - C_2^2)^2 + 4C_3^2}$.

The inversion of (5.25) is next. From (5.48), figure 1 and by (6.2), (6.5), (6.11) to (6.14) we conclude that for $t > 0$,

$$I_{42}(R,t) + J_7(R,t) + J_8(R,t) + J_9(R,t) =$$

$$\text{Res } \{\bar{I}_{42}(R,p)e^{pt} \text{ where } C_4^2 p + v_2^2 = u^2 \text{ and } u = 0\}. \tag{6.24}$$

The terms J_7 to J_9 are given by

$$J_7(R,t) = -\frac{1}{C_4^2\pi}\int_0^\infty \frac{2(C_2^2 - v_2^2 - r) - F_1(r)}{F_1(r)\ r_1(r)} \sin\left[\frac{(r+v_2^2)R}{C_1C_4^2} \left(\frac{\Gamma_2(r)}{2r}\right)^{1/2}\right]$$

$$\cdot \exp\left\{ \frac{-t(r+v_2^2)}{C_4^2} \right\} dr \tag{6.25}$$

according to (6.2);

$J_8(R,t) =$

$$-\frac{1}{\pi C_4^2}\ \mathrm{Re}\int_0^1 \frac{C_1^2 + 2iC_3\omega - \Phi_1(\omega)e^{i\theta_1}}{\sqrt{1-\omega^2}\ \Phi_1(\omega)^{i\theta_1}}$$

$$\cdot \exp\left[\frac{C_1^2+C_2^2-2iC_3\omega}{C_4^2}\ \left(t - \frac{R\ e^{\frac{-i\theta_1}{2}}}{\Phi_1^{1/2}(\omega)}\right)\right] d\omega \tag{6.26}$$

and

$J_9(R,t) =$

$$-\frac{1}{\pi C_4^2}\ \mathrm{Re}\int_0^1 \frac{C_1^2 + 2iC_3\omega - \Phi_2(\omega)e^{i\theta_2}}{\sqrt{1-\omega^2}\ \ \Phi_2(\omega)e^{i\theta_2}}$$

$$\cdot \exp\left[\frac{C_1^2-C_2^2+2iC_3\omega}{C_4^2}\ \left(t - \frac{R\ e^{-\frac{i\theta_2}{2}}}{\Phi_2^{1/2}(\omega)}\right)\right] d\omega \tag{6.27}$$

in each of which (6.12) to (6.14) have been used.

The inversion is continued and we examine the terms $\bar{I}_{31}$, $\bar{I}_{32}$ and $\bar{I}_{33}$ defined in section 5. We begin by using contour C_1 of figure 1 and by recalling (5.28). Thus, when $t < R/C_1$, then,

$$\lim_{R_3 \to \infty} \frac{1}{2\pi i} \oint_{C_1} [\bar{I}_{31}(R,p),\ \bar{I}_{32}(R,p),\ \bar{I}_{33}(R,p)]e^{pt}dp = 0 \qquad (6.28)$$

by the Cauchy integral theorem. Inversion of these quantities yield non-zero values only for $t > R/C_1$.

The actual inversion is more easily accomplished if we recall that the definition of the $\bar{I}_{3j}$, $j = 1,2,3$ quantities are obtained from the $\bar{I}_{4j}$'s by replacing $F_1(p)$ by $F_2(p)$ and $F_2(p)$ by $F_1(p)$ in them. This change, especially in the exponential factors, eliminates the contributions along the arcs DE, EWW′F and FG of figure 1. Thus for $\bar{I}_{3j}$, $j = 1,2,3$, the inversion for $t > R/C_1$ is given by

$$I_{3j}(R,t) + \frac{1}{2\pi i}\left[\int_{TM} + \int_{NP} + \int_{PQ} + \int_{RS}\right] \bar{I}_{3j}(R,p)e^{pt}dp$$

$$= \text{Res}\ \{\bar{I}_{3j}(R,p)e^{pt} \text{ at any poles inside } C\}. \qquad (6.29)$$

In particular, for $j = 1$, by (6.12) to (6.16) and (6.28),

$$I_{31}(R,t) + J_2(R,t) + J_3(R,t) = \text{Res}\{\bar{I}_{31}(R,p)e^{pt} \text{ at } p = 0\} \qquad (6.30)$$

for $t > R/C_1$. In fact for all $t > 0$

$$I_{31}(R,t) = -[J_2(R,t) + J_3(R,t)]H(t-R/C_1)$$

$$+ \text{Res}\ [\bar{I}_{31}(R,p)e^{pt} \text{ at } p = 0]\ H(t-R/C_1) \qquad (6.31)$$

where $H(\ )$ is the Heaviside step function.

Similarly, by (6.21), (6.22), (6.26) and (6.27).

$$I_{32}(R,t) = -[J_8(R,t) + J_9(R,t)]\, H(t-{}^{R}/_{C_1}) \tag{6.32}$$

and

$$I_{33}(R,t) = -[J_5(R,t) + J_6(R,t)]\, H(t-{}^{R}/_{C_1})$$
$$+ \operatorname{Res}[\bar{I}_{33}(R,p)e^{pt} \text{ at } p=0]\; H(t-{}^{R}/_{C_1}). \tag{6.33}$$

An examination of (5.22) shows that all terms necessary to invert $\hat{w}_m$ have now been inverted with the exception of $\bar{I}_{m\nu}(R,p)$ given in (5.23). But, by inspection, the inverse of $\bar{I}_{m\nu}$ is clearly given by

$$I_{m\nu}(R,t) = \frac{\delta_{m1}t}{4\pi\Omega}\frac{\partial}{\partial\Omega}\left[\frac{\Omega^2}{R^3}H(t-\frac{R}{v_s})\right]$$
$$+ \frac{\delta_{m\nu}}{4\pi}\frac{\partial^2}{\partial x_\nu \partial x_1}\left[\frac{1}{R}(t-\frac{R}{v_s})\, H(t-\frac{R}{v_s})\right] \tag{6.34}$$

for $t > 0$.

We can now collect all of these pieces together and write an expression for $w_m(x_1,x_2,x_3,t)$ for all $t > 0$. We use (6.17), (6.19), (6.24), (6.31), (6.32), (6.33) and (6.34) to give for $t > 0$

$$w_m(x_1,x_2,x_3,t) = -\frac{\delta_{m1}t}{4\pi\Omega}\frac{\partial}{\partial\Omega}\left[\frac{\Omega^2}{R^3}H(t-\frac{R}{v_s})\right]$$
$$-\frac{\delta_{m\nu}}{4\pi}\frac{\partial^2}{\partial x_\nu \partial x_1}\left[\frac{1}{R}(t-\frac{R}{v_s})H(t-\frac{R}{v_s})\right]$$
$$+\frac{\delta_{m1}(C_1^2-v_s^2)}{4\pi\Omega}\frac{\partial}{\partial\Omega}\left[\frac{\Omega^2}{R^2}\{-J_1(R,t)H(t) - J_2(R,t)H(\frac{R}{C_1}-t)\right.$$
$$\left. - J_3(R,t)H(\frac{R}{C_1}-t) + K_{41}(R,t)H(t) + K_{42}(R,t)H(t) - K_{31}(R,t)H(t-\frac{R}{C_1})\}\right]$$

$$- \frac{\delta_{m1}}{4\pi r} \{-J_7(R,t)H(t) - J_8(R,t)H(\frac{R}{C_1} - t) - J_9(R,t)H(\frac{R}{C_1} - t)$$

$$+ K_{45}(R,t)H(t)\} \qquad (6.35)$$

$$+ \frac{\delta_{m\nu}(C_1^2 - v_s^2)}{4\pi} \frac{\partial^2}{\partial x_\nu{}^2 x_1} \Big[\frac{1}{R}\{- J_4(R,t)H(t) - J_5(R,t)H(\frac{R}{C_1} - t)$$

$$- J_6(R,t)H(\frac{R}{C_1} - t) + K_{43}(R,t)H(t) + K_{44}(R,t)H(t)$$

$$- K_{33}(R,t)H(t - \frac{R}{C_1})\}\Big]$$

for $m = 1,2,3$ and $\nu = 2,3$.

The $K_{\alpha\beta}(R,t)$ introduced here correspond to the residue terms of the various $\bar{I}_{\alpha\beta}$ as follows:

$$K_{41}(R,t) = \mathrm{Res}[\bar{I}_{41}(R,p)e^{pt} \text{ at } p=0],$$

$$K_{42}(R,t) = \mathrm{Res}[\bar{I}_{41}(R,u)e^{\frac{t(u^2-v_2^2)}{C_4^2}} \text{ at } u = 0] \text{ defined in (6.10)},$$

$$K_{31}(R,t) = \mathrm{Res}[\bar{I}_{31}(R,p)e^{pt} \text{ at } p = 0],$$

$$K_{43}(R,t) = \mathrm{Res}[\bar{I}_{43}(R,p)e^{pt} \text{ at } p = 0], \qquad (6.36)$$

$$K_{44}(R,t) = \mathrm{Res}[\bar{I}_{43}(R,u)e^{\frac{t(u^2-v_2^2)}{C_4^2}} \text{ at } u = 0].$$

$$K_{45}(R,t) = \mathrm{Res}[\bar{I}_{42}(R,u)e^{\frac{t(u^2-v_2^2)}{C_4^2}} \text{ at } u = 0],$$

$$K_{33}(R,t) = \mathrm{Res}[\bar{I}_{33}(R,p)e^{pt} \text{ at } p = 0].$$

Equations (6.35) and (6.36) give the completed inversion of the solid displacement components in terms of integrals and infinite series.

By the same procedures used to accomplish the inversion of $\hat{w}_m(x_1,x_2,x_3,p)$ given in (5.22) we can also invert the expression for $\hat{v}_m$ given in (4.19) for the case when $\alpha = 0$.

When the value $\alpha = 0$ is used in (4.18) and (4.19), and when (5.1) to (5.3), (5.7) to (5.16), and (5.21) are applied, then $\hat{v}_m(x_1,x_2,x_3,p)$ may be written as

$$\hat{v}_m(x_1,x_2,x_3,p) = \frac{C_3\delta_{m1}(\bar{\rho}_1/\bar{\rho}_2)^{1/2}}{2\pi R}\left[\bar{L}_{21}(R,p) - \bar{L}_{11}(R,p)\right]$$
$$+ \frac{C_3\delta_{m1}(\bar{\rho}_1/\bar{\rho}_2)^{1/2}}{4\pi\Omega}\frac{\partial}{\partial\Omega}\left[\frac{\Omega^2}{R^2}\{\bar{L}_{22}(R,p) - \bar{L}_{12}(R,p)\}\right] \quad (6.37)$$
$$- \frac{C_3\delta_{m\nu}(\bar{\rho}_1/\bar{\rho}_2)^{1/2}}{4\pi}\frac{\partial^2}{\partial x_\nu \partial x_1}\left[\frac{\bar{L}_{23}(R,p) - \bar{L}_{13}(R,p)}{R}\right],$$

for $m = 1,2,3$, $\nu = 2,3$.

The terms $\bar{L}_{\alpha\beta}(R,p)$ are

$$\bar{L}_{21}(R,p) = \frac{p}{F_2(p)\{[C_4^2p-(C_1^2-C_2^2)]^2 + 4C_3^2\}^{1/2}}\exp\left\{\frac{-\sqrt{2}\,p\,R}{F_2^{1/2}(p)}\right\},$$

$$\bar{L}_{22}(R,p) = \frac{1}{p\{[C_4^2p-(C_1^2-C_2^2)]^2 + 4C_3^2\}^{1/2}}\left[\frac{\sqrt{2}\,p}{F_2^{1/2}(p)} + \frac{1}{R}\right]$$
$$\cdot \exp\left\{\frac{-\sqrt{2}\,p\,R}{F_2^{1/2}(p)}\right\}, \quad (6.38)$$

$$\bar{L}_{23}(R,p) = \frac{1}{p\{[C_4^2p-(C_1^2-C_2^2)]^2 + 4C_3^2\}^{1/2}}\exp\left\{\frac{-\sqrt{2}\,p\,R}{F_2^{1/2}(p)}\right\},$$

with $\bar{L}_{1\beta}(R,p)$ obtained from $\bar{L}_{2\beta}(R,p)$ by replacing $F_2(p)$ by $F_1(p)$ wherever it appears.

The inversion of the $\bar{L}_{\alpha\beta}(R,p)$ is so similar to the inversion of the $\bar{I}$'s that only the final expressions are given. Therefore, for $t > 0$,

$$\left.\begin{aligned}
&L_{11}(R,t) + \mathfrak{M}_1(R,t) + \mathfrak{M}_2(R,t) + \mathfrak{M}_3(R,t) = Q_{11}(R,t),\\
&L_{12}(R,t) + \mathfrak{M}_4(R,t) + \mathfrak{M}_5(R,t) + \mathfrak{M}_6(R,t) = Q_{12}(R,t) + Q_{13}(R,t),\\
&L_{13}(R,t) + \mathfrak{M}_7(R,t) + \mathfrak{M}_8(R,t) + \mathfrak{M}_9(R,t) = Q_{14}(R,t) + Q_{15}(R,t),\\
&L_{21}(R,t) = -[\mathfrak{M}_2(R,t) + \mathfrak{M}_3(R,t)]\,H(t - R/C_1),\\
&L_{22}(R,t) = -[\mathfrak{M}_5(R,t) + \mathfrak{M}_6(R,t) - Q_{13}(R,t)]\,H(t - R/C_1),
\end{aligned}\right\}\quad(6.39)$$

$$L_{23}(R,t) = -[\mathfrak{M}_8(R,t) + \mathfrak{M}_9(R,t) - Q_{15}(R,t)]\,H(t - R/C_1),$$

where, by (6.2),

$$\left.\begin{aligned}
&\mathfrak{M}_1(R,t) = -\frac{1}{\pi\, C_4^4}\int_0^\infty \frac{r + v_2^2}{r_1(r)\Gamma_1(r)}\sin\{E_3(r,R)\}\exp\left\{\frac{-t(r+v_2^2)}{C_4^2}\right\}dr,\\
&\mathfrak{M}_4(R,t) =\\
&\quad\frac{1}{\pi R}\int_0^\infty \frac{1}{r_1(r)[r+v_2^2]}\sin\{E_3(r,R)\}\exp\left\{\frac{-t(r+v_2^2)}{C_4^2}\right\}dr\\
&\quad-\frac{1}{C_1C_4^2\pi}\int_0^\infty \frac{1}{r_1(r)}\left(\frac{\Gamma_1(r)}{2r}\right)^{1/2}\cos\{E_3(r,R)\}\exp\left\{\frac{-t(r+v_2^2)}{C_4^2}\right\}dr,\\
&\mathfrak{M}_7(R,t) = \frac{1}{\pi}\int_0^\infty \frac{1}{r_1(r)[r+v_2^2]}\sin\{E_3(r,R)\}\exp\left\{\frac{-t(r+v_2^2)}{C_4^2}\right\}dr,
\end{aligned}\right\}\quad(6.40)$$

with

$$E_3(r,R) = \frac{(r+v_2^2)R}{C_1C_4^2}\left(\frac{\Gamma_1(r)}{2r}\right)^{1/2}.$$

From (6.12) to (6.14), the remanining $\mathfrak{M}$'s are

$$\mathfrak{M}_2(R,t) = -\frac{1}{2\pi C_4^4} \operatorname{Re} \int_0^1 E_4(\omega) \exp\{E_1(\omega,R,t) - i\theta_1\} d\omega,$$

$$\mathfrak{M}_3(R,t) = -\frac{1}{2\pi C_4^4} \operatorname{Re} \int_0^1 E_5(\omega) \exp\{E_2(\omega,R,t) - i\theta_2\} d\omega,$$

$$\mathfrak{M}_5(R,t) = -\frac{1}{\pi} \operatorname{Re} \int_0^1 (1-\omega^2)^{-1/2} \left[\frac{E_6(\omega)}{R} + \frac{e^{\frac{-i\theta_1}{2}}}{C_4^2 \Phi_1^{1/2}(\omega)}\right] \exp\{E_1(\omega,R,t)\} d\omega, \tag{6.41}$$

$$\mathfrak{M}_6(R,t) = -\frac{1}{\pi} \operatorname{Re} \int_0^1 (1-\omega^2)^{-1/2} \left[\frac{E_6(\omega)}{R} + \frac{e^{\frac{-i\theta_2}{2}}}{C_4^2 \Phi_2^{1/2}(\omega)}\right] \exp\{E_2(\omega,R,t)\} d\omega,$$

$$\mathfrak{M}_8(R,t) = -\frac{1}{\pi} \operatorname{Re} \int_0^1 (1-\omega^2)^{-1/2} E_6(\omega) \exp\{E_1(\omega,R,t)\} d\omega,$$

$$\mathfrak{M}_9(R,t) = -\frac{1}{\pi} \operatorname{Re} \int_0^1 (1-\omega^2)^{-1/2} E_6(\omega) \exp\{E_2(\omega,R,t)\} d\omega.$$

The functions E_k used here may be defined as

$$E_1(\omega,R,t) = \frac{C_1^2 - C_2^2 + 2iC_3\omega}{C_4^2} \left[t - R\Phi_1^{-1/2}(\omega) \exp\left\{\frac{-i\theta_1}{2}\right\}\right],$$

$$E_2(\omega,R,t) = \frac{C_1^2 - C_2^2 + 2iC_3\omega}{C_4^2} \left[t - R\Phi_2^{-1/2}(\omega) \exp\left\{\frac{-i\theta_2}{2}\right\}\right],$$

$$E_4(\omega) = \frac{C_1^2-C_2^2+2iC_3\omega}{\Phi_1(\omega)}(1-\omega^2)^{-1/2}, \quad E_5(\omega) = \frac{C_1^2-C_2^2+2iC_3\omega}{\Phi_2(\omega)}(1-\omega^2)^{-1/2},$$

$$E_6(\omega) = \frac{C_1^2-C_2^2-2iC_3\omega}{(C_1^2-C_2^2)^2+4C_3^2\omega^2} .$$

The Q terms represent the residues of the various $\bar{L}_{\alpha\beta}$ and are defined below. Using (6.5) and (6.11),

$$Q_{11}(R,t) = \text{Res}\left[\frac{(u^2-v_2^2)\ F_2(u)}{C_1^2(C_4^2)^2\ u\ r_1\ (u)} \exp\left\{\frac{u^2-v_2^2}{C_4^2}\left[t-\frac{RF_2^{1/2}(u)}{\sqrt{2}\ C_1\ u}\right]\right\} \text{at } u=0\right],$$

$$Q_{12}(R,t) = \text{Res}\left[\frac{2u}{r_1(u)[u^2-v_2^2]}\left\{\frac{(u^2-v_2^2)F_2^{1/2}(u)}{\sqrt{2}\ C_1C_4^2\ u}+\frac{1}{R}\right\}\right.$$

$$\left.\cdot \exp\left\{\frac{u^2-v_2^2}{C_4^2}\left[t-\frac{RF_2^{1/2}(u)}{\sqrt{2}\ C_1\ u}\right]\right\} \text{ at } u=0\right], \tag{6.42}$$

$$Q_{14}(R,t) = \text{Res}\left[\frac{2u}{r_1(u)[u^2-v_2^2]} \exp\left\{\frac{u^2-v_2^2}{C_4^2}\left[t-\frac{RF_2^{1/2}(u)}{\sqrt{2}\ C_1\ u}\right]\right\} \text{ at } u=0\right],$$

$$Q_{13}(R,t) = \frac{-1}{R\sqrt{(C_1^2-C_2^2)^2+4C_3^2}},$$

$$Q_{15}(R,t) = \frac{-1}{\sqrt{(C_1^2-C_2^2)^2+4C_3^2}} .$$

Gathering all of this together, i.e., (6.37) to (6.42) we have for $t > 0$,

$$v_m(x_1,x_2,x_3,t) =$$

$$- \frac{C_3\delta_{m1}}{2\pi R}\left(\frac{\bar{\rho}}{\bar{\bar{\rho}}_2}\right)^{1/2} [\, -\mathfrak{M}_1(R,t)H(t) - \mathfrak{M}_2(R,t)H(\frac{R}{C_1}-t) - \mathfrak{M}_3(R,t)H(\frac{R}{C_1}-t)$$

$$+ Q_{11}(R,t)H(t)\,]$$

$$- \frac{C_3\delta_{m1}}{4\pi\Omega}\left(\frac{\bar{\rho}_1}{\bar{\bar{\rho}}_2}\right)^{1/2}\frac{\partial}{\partial\Omega}\Big[\frac{\Omega^2}{R^2}\{ -\mathfrak{M}_4(R,t)H(t) - \mathfrak{M}_5(R,t)H(\frac{R}{C_1}-t)$$

$$- \mathfrak{M}_6(R,t)H(\frac{R}{C_1}-t) + Q_{12}(R,t)H(t) + Q_{13}(R,t)H(\frac{R}{C_1}-t)\}\Big] \tag{6.43}$$

$$+ \frac{C_3\delta_{m\nu}}{4\pi}\left(\frac{\bar{\rho}_1}{\bar{\bar{\rho}}_2}\right)^{1/2}\frac{\partial^2}{\partial x_\nu \partial x_1}\Big[\frac{1}{R}\{-\mathfrak{M}_7(R,t)H(t) - \mathfrak{M}_8(R,t)H(\frac{R}{C_1}-t)$$

$$- \mathfrak{M}_9(R,t)H(\frac{R}{C_1}-t) + Q_{14}(R,t)H(t) + Q_{15}(R,t)H(\frac{R}{C_1}-t)\}\Big]$$

for $m = 1,2,3$ and $\nu = 2,3$.

Expressions (6.35) and (6.43) constitute the solution of the title problem of this paper for the case when $\alpha = 0$. To give these expressions in terms of the original coordinates we employ the coordinate shift, given in (2.12), to (6.35) and (6.43). The apparent changes in (6.35) and (6.43) are to replace the x_i, Ω and R by $x_i - x_{i_0}$, Ω_0, R_0, respectively,

where

$$x_{i_0} = (x_0,\ y_0,\ z_0)$$

$$\Omega_0 = \sqrt{(x_2-y_0)^2 + (x_3-z_0)^2}$$

$$R_0 = \sqrt{(x_1-x_0)^2 + \Omega_0^2}$$

As mentioned previously, theoretically the real integral representations presented here constitute the exact solution. However, as they stand, none of these integrals are easily evaluated - analytically or numerically - unless specific ranges of values of the constants (5.9) or (5.13) are known. Experimental data for mixture theories, such as the one used here, have not been published to the best of our knowledge and the process of piecing together material properties obtained for individual solids and fluids and attempting to fit these data to the constitutive equations is, at best, a "lucky" guess.

Another approach, satisfactory mathematically, is to make asymptotic estimates of these integrals by taking combinations of the space and time variables and evaluating the integrals by means of a variety of methods.

Alternatively, we could further idealize the materials composing the mixture in the hope of obtaining solutions more readily applicable. This is done in section 7.

7. Special cases. In this section we present the solution to the problem of section 2, subject to the condition $\alpha = 0$ and to additional restrictions.

A. Fluid inviscid

If the fluid is inviscid then by the constitutive equations (2.4)

$$\pi_{ij} = -\alpha_1 \delta_{ij} \tag{7.1}$$

and we set

$$\mu = \gamma = \gamma_1 = \gamma_2 = 0 \, . \tag{7.2}$$

Note that $\mu = \gamma = 0$ are the usual assumptions of inviscid fluids and the additional $\gamma_1 = \gamma_2 = 0$ frees the fluid partial stress from dependence upon the solid strain and fluid density. Assumption (7.2) when put into (2.11) requires us to set

$$K_2 = 0 \quad , \quad \bar{\rho}_1\theta_1 + \bar{\rho}_2\theta_2 = 0 \, . \tag{7.3}$$

When (7.3) is used in (5.9), (5.12), (5.13) and (5.21), we obtain

$$\left.\begin{aligned}
&c_4^2 = 0, \\
&\left\{[c_4^2 p - (c_1^2-c_2^2)]^2 + 4c_3^2\right\}^{1/2} \equiv \sqrt{(c_2^2-c_1^2)^2 + 4c_3^2} \equiv r_1, \\
&F_1(p) \equiv F_1 \equiv c_1^2 + c_2^2 + r_1, \\
&F_2(p) \equiv F_2 \equiv c_1^2 + c_2^2 - r_1.
\end{aligned}\right\} \tag{7.4}$$

The direct consequence of (7.2) and (7.4) is that the expressions (5.22) and (6.37) for $\hat{w}_m$ and $\hat{v}_m$ contain only poles at $p = 0$ and the inverses can be calculated easily. Without writing the intermediate steps it is very easy to see that the displacement and the velocity are given by (recall (2.12))

$$\left.\begin{aligned}
&w_1(\vec{x},\vec{x}_0,t) = \frac{t}{4\pi R_0{}^3} \; G_1(\vec{x},\vec{x}_0,t), \\
&w_\nu(\vec{x},\vec{x}_0,t) = -\frac{(x_1-x_0)(x_\nu-x_{\nu 0})t}{4\pi \, R_0{}^4} \; G_2\;(\vec{x},\vec{x}_0,t),
\end{aligned}\right\} \tag{7.5}$$

$$\left.\begin{aligned} v_1(\vec{x},\vec{x}_0,t) &= \frac{C_3}{4\pi r_1}\left(\frac{\bar{\rho}_1}{\bar{\rho}_2}\right)^{1/2}\frac{1}{R_0{}^2}\ G_3(\vec{x},\vec{x}_0,t), \\ v_\nu(\vec{x},\vec{x}_0,t) &= \frac{C_3}{4\pi\ R_0{}^3\ r_1}\left(\frac{\bar{\rho}_1}{\bar{\rho}_2}\right)^{1/2}(x_1-x_0)\,(x_\nu-x_{\nu 0})\ G_4(\vec{x},\vec{x}_0.t), \end{aligned}\right\} \tag{7.5}$$

$$x_{\nu 0} = \delta_{\nu 2}y_0 + \delta_{\nu 3}z_0 \qquad R_0 = \sqrt{(x_1-x_0)^2 + (x_2-y_0)^2 + (x_3-z_0)^2},$$

$$\nu = 2,3,$$

where

$$\begin{aligned} G_1(\vec{x},\vec{x}_0,t) = {} & [1 - \frac{3(x_1-x_0)^2}{R_0{}^2}]\ H(t - \frac{R_0}{v_s}) + \frac{\Omega_0^2}{R_0 v_s}\ \delta(t-\frac{R_0}{v_s}) \\ & + \frac{C_1^2-v_s^2}{r_1}\Big[\frac{2(v_1^2+v_s^2)-F_2}{2v_s^2 - F_2}\Big\{[1 - \frac{3(x_1-x_0)^2}{R_0{}^2}]\ H(t - \frac{\sqrt{2}\ R_0}{F_2{}^{1/2}}) \\ & \qquad - \frac{\sqrt{2}\ (x_1-x_0)^2}{R_0\ F_2{}^{1/2}}\ \delta(t - \frac{\sqrt{2}\ R_0}{F_2{}^{1/2}})\Big\} \\ & - \frac{2(v_1^2+v_s^2)-F_1}{2v_s^2-F_1}\Big\{[1 - \frac{3(x_1-x_0)^2}{R_0{}^2}]\ H(t - \frac{\sqrt{2}\ R_0}{F_1{}^{1/2}}) \\ & \qquad - \frac{\sqrt{2}(x_1-x_0)^2}{R_0F_1{}^{1/2}}\ \delta(t - \frac{\sqrt{2}\ R_0}{F_1{}^{1/2}})\Big\}\Big], \end{aligned} \tag{7.6}$$

$$\begin{aligned} G_2(\vec{x},\vec{x}_0,t) = {} & \frac{3}{R_0}\ H(t - \frac{R_0}{v_s}) + \frac{1}{v_s}\ \delta(t - \frac{R_0}{v_s}) \\ & + \frac{C_1^2-v_s^2}{r_1}\Big[\frac{2(v_1^2+v_s^2)-F_2}{2v_s^2 - F_2}\Big\{\frac{3}{R_0}H(t - \frac{\sqrt{2}\ R_0}{F_2{}^{1/2}}) + \frac{\sqrt{2}}{F_2{}^{1/2}}\delta(t - \frac{\sqrt{2}\ R_0}{F_2{}^{1/2}})\Big\} \\ & - \frac{2(v_1^2-v_s^2)-F_1}{2v_s^2 - F_1}\Big\{\frac{3}{R_0}\ H(t - \frac{\sqrt{2}\ R_0}{F_1{}^{1/2}}) + \frac{\sqrt{2}}{F_1{}^{1/2}}\ \delta(t - \frac{\sqrt{2}\ R_0}{F_1{}^{1/2}})\Big\}\Big] \end{aligned} \tag{7.7}$$

$$
\begin{aligned}
G_3(\vec{x},\vec{x}_0,t) &= \frac{1}{R_0}\left[\frac{3(x_1-x_0)^2}{R_0^{\,2}} - 1\right] H\left(t - \frac{\sqrt{2}\,R_0}{F_2^{\,1/2}}\right) \\
&\quad + \frac{\sqrt{2}}{F_2^{\,1/2}}\left[\frac{3(x_1-x_0)^2}{R_0^{\,2}} - 1\right]\delta\left(t - \frac{\sqrt{2}\,R_0}{F_2^{\,1/2}}\right) \\
&\quad + \frac{2(x_1-x_0)^2}{R_0F_2}\,\delta'\left(t - \frac{\sqrt{2}\,R_0}{F_2^{\,1/2}}\right) \\
&\quad - \frac{1}{R_0}\left[\frac{3(x_1-x_0)^2}{R_0^{\,2}} - 1\right] H\left(t - \frac{\sqrt{2}\,R_0}{F_1^{\,1/2}}\right) \\
&\quad - \frac{\sqrt{2}}{F_1^{\,1/2}}\left[\frac{3(x_1-x_0)^2}{R_0^{\,2}} - 1\right]\delta\left(t - \frac{\sqrt{2}\,R_0}{F_1^{\,1/2}}\right) \\
&\quad - \frac{2(x_1-x_0)^2}{R_0F_1}\,\delta'\left(t - \frac{\sqrt{2}\,R_0}{F_1^{\,1/2}}\right),
\end{aligned}
\tag{7.8}
$$

$$
\begin{aligned}
G_4(\vec{x},\vec{x}_0,t) &= \frac{3}{R_0^{\,2}}\left[H\left(t - \frac{\sqrt{2}\,R_0}{F_1^{\,1/2}}\right) - H\left(t - \frac{\sqrt{2}\,R_0}{F_2^{\,1/2}}\right)\right] \\
&\quad + \frac{3}{R_0}\left[\frac{\sqrt{2}}{F_1^{\,1/2}}\,\delta\left(t - \frac{\sqrt{2}\,R_0}{F_1^{\,1/2}}\right) - \frac{\sqrt{2}}{F_2^{\,1/2}}\,\delta\left(t - \frac{\sqrt{2}\,R_0}{F_2^{\,1/2}}\right)\right] \\
&\quad + \frac{2}{F_1}\,\delta'\left(t - \frac{\sqrt{2}\,R_0}{F_1^{\,1/2}}\right) - \frac{2}{F_2}\,\delta'\left(t - \frac{\sqrt{2}\,R_0}{F_2^{\,1/2}}\right).
\end{aligned}
\tag{7.9}
$$

The notation $\delta'(x)$ means $\frac{d}{dx}\delta(x)$ in (7.8) and (7.9).

Equations (7.5) to (7.9) represent the complete solution of the problem of a point force impulsively applied to the solid component along a direction parallel to the x_1 axis at a point $\vec{x}_0$ for the case when the diffusive resistance of the mixture is zero and for which the fluid is inviscid. Several properties of the solution are noted.

The solution is symmetric in $\vec{x}$ and $\vec{x}_0$. The response at $\vec{x}$ due to the force at $\vec{x}_0$ is the same as the response at $\vec{x}_0$ due to the force applied at $\vec{x}$. Hence,

$$\begin{aligned} \vec{w}(\vec{x},\vec{x}_0,t) &= \vec{w}(\vec{x}_0,\vec{x},t) \\ \vec{v}(\vec{x},\vec{x}_0,t) &= \vec{v}(\vec{x}_0,\vec{x},t). \end{aligned} \tag{7.10}$$

If, instead of the force specified in (2.7), we used

$$\begin{aligned} \vec{f} &= \vec{a}_y \ \delta(\vec{x}-\vec{x}_0)\delta(t) \\ \vec{g} &= \vec{0} \end{aligned} \tag{7.11}$$

then the solution of this problem is obtained from (7.5) to (7.9) by interchanging w_1 and w_2, v_1 and v_2, x_1 and x_2, x_0 and y_0. Carrying this thought one step further, the solution to the problem in which the body force is a linear combination of point loads in an arbitrary direction is a linear combination of solutions such as the one derived here, the one corresponding to (7.11) and one corresponding to the loading

$$\begin{aligned} \vec{f} &= \vec{a}_z \ \delta(\vec{x}-\vec{x}_0) \ \delta(t) \\ \vec{g} &= \vec{0}. \end{aligned}$$

When the force is applied to the fluid component instead of the solid, the methods of this paper may be applied unchanged to derive the solution to such a problem.

Finally, if the fluid component were completely absent and only the solid occupies the region, then the solution (7.5) to (7.9) becomes the solution for the response of an elastic body to a point impulsive load. This solution has been given by Payton [6].

B. <u>Uncoupled field equations</u>.

If, in addition to $\alpha = 0$, we set $\Theta_1 = 0$ in the field equations (2.10), they become uncoupled and the fluid velocity response is

$$v_m(\vec{x},t) \equiv 0 \qquad m = 1,2,3.$$

The effect of $\alpha = \Theta_1 = 0$ in (2.10) is to reduce them to the elastic equations of motion if one identifies β_3 with the elastic shear modulus while taking β_2 to be the other Lamé constant.

The solution (7.5) to (7.9) becomes the elastic response if in these equations we identify v_s and c_1 as elastic equivoluminal and irrotational wave velocities, respectively, and set $c_2 = c_3 = 0$.